Preface

This book contains an introduction to the mathematical theory of financial markets with proportional transaction costs. Traditionally, a theoretical analysis of models with market imperfections was considered as the most challenging and difficult chapter of mathematical finance. Nowadays there are hundreds of papers on this subject, but it still is not covered by monographic literature fixing the main achievements. We propose here a highly subjective selection of results, which, as we hope, give an idea what is going on and which may serve as a platform for further studies. The main topics are: approximative hedging, arbitrage theory, and consumption–investment problems.

Our interest in the subject originates from the famous paper of Heyne Leland, who suggested a method how to price contingent claims on a market with constant proportional transaction costs and gave to the traders a practically important benchmark. From the economical perspectives his idea is to replace the classical Black–Scholes principle of "pricing by replication" (coinciding, in the case of complete markets, with the principle of "pricing by arbitrage") by the principle of "pricing by an approximative hedging." This approximative hedging can be realized in various ways. Leland's suggestion was to apply the common algorithm of periodical portfolio revisions using the common Black–Scholes formulae but with an appropriately enlarged volatility. This method happened to be very efficient for practical values of the model parameters where the transaction costs are small. It is widely accepted in the financial industry. However, a more detailed analysis shows some interesting mathematical aspects of this approach: even for the standard call option, the terminal value of the replicating portfolio does not converge to the terminal pay-off if the transaction cost coefficients do not depend on the number of revisions (tending to infinity). The limiting discrepancy can be calculated explicitly. This is a rigorous mathematical result disclaiming but not discarding Leland's approach. In his paper Leland conjectured that the convergence takes place when the transaction costs are decreasing to zero as $n^{-1/2}$, where n is the number of revisions. This was confirmed in the thesis of Klaus Lott, who provided necessary mathematical arguments; we refer to this particular case where the enlarged volatility does not depend on n as to the Leland–Lott model. In fact, the approximation error always tends to zero if the transaction costs tend to zero (with any rate). The latter property explains the applicability of the Leland idea and its importance for the practical purposes: the trader may assume that the real-world market is described by a model with a particular but sufficiently large number n of revision intervals for which the transaction cost coefficients k_n are small enough.

The case investigated by Lott seems to be the most important. This is the reason why we concentrate our efforts on the analysis of the asymptotical behavior of the hedging error. We obtain the exact rate of asymptotics of the L^2-norm of this error for a large class of options with convex pay-off functions. We consider the setting with nonuniform revision intervals and establish an asymptotic expansion when the revision dates are $t_i^n = g(i/n)$, where the strictly increasing scale function $g : [0,1] \to [0,1]$ and its inverse f are continuous with their first and second derivatives on the whole interval or $g(t) = 1-(1-t)^\beta$, $\beta \geq 1$. We show that the sequence $n^{1/2}(V_T^n - V_T)$ converges in law to a random variable which is the terminal value of a component of a two-dimensional Markov diffusion process and calculate the limit.

It is worth noticing that the result that there is no convergence in the case where the transaction costs do not depend on the model number also has some practical implications: the revealed structure of the discrepancy explains the empirical fact that the precision of approximation is worse in the case where the stock at the maturity date evolves near the pay-off. We discuss this and other aspects of the Leland strategy in our first chapter. In particular, we provide a formulation of the Pergamenshchikov limit theorem which gives a limiting distribution of the approximation error corrected by the discrepancy.

We present also the beautiful proof due to Skorokhod and Levental of the Davis–Norman conjecture that in the presence of market friction the hedging (super-replication) of a call option on the stock evolving as a geometric Brownian motion can be achieved by a single buy-and-hold transaction, at the beginning of the trading period, without further trading.

In the second chapter we develop an arbitrage theory for financial markets without friction.

First, we recall the classical arbitrage theory for frictionless market models in discrete time, providing a self-contained and rather exhaustive synthesis of the known results. We give a detailed analysis of the Dalang–Morton–Willinger theorem in its modern formulation, which is a list of equivalent conditions, and show that for the model with restricted information, the list is necessarily shorter. Our presentation is adapted for the treatment of more delicate problems for transaction cost model. Note that the mentioned classical topic is rarely discussed in textbooks on mathematical finance being considered as too complicated. By this reason we present a “fast” and “elementary” proof of the major equivalence suitable for lecture courses. It is based on a combination of the original approach due to Chris Rogers with a lemma on convergent measurable subsequences.

We also discuss the structure of equivalent martingale measures and prove the theorem that in the case where the reference measure is a martingale one, the martingale measures with bounded densities are norm-dense in the set of all martingale measures. This implies, in particular, that the set of martingale measures with finite entropy, if nonempty, is dense in the set of all martingale measures. This section also contains a simple proof of the optional decomposition theorem, which is, in the discrete-time setting, a very simple result. We also prove a hedging theorem for European options, which asserts that the set of initial endowments for (self-financing) portfolios super-replicating a given contingent claim is a closed interval. Its left extremity is the supremum of expectations of the contingent claim with respect to the set of all martingale measures. We go beyond finite-horizon setting and prove some no-arbitrage criteria for infinite-horizon models. We conclude the section by an example of application of the duality theory to a utility maximization problem and a brief comment on continuous-time models.

With the above preliminaries, in the third chapter we attack the problem of no-arbitrage conditions for markets with proportional transaction costs.

As a mathematical description for the latter, we use a general scheme of two adapted cone-valued processes in convex duality, giving, at least for mathematicians, a comprehensive "parameter-free" description of the main objects of the theory. In the financial context the values of the primary processes are polyhedral cones, describing solvency regions (evolving in time and depending on the state of the nature). The portfolio processes are vector-valued; they can be viewed either in terms of quotes (i.e., units of a certain numéraire) or in terms of "physical units" (e.g., for models of currency markets, positions in euros, dollars, yens, etc.); both descriptions are related in the obvious way. It is convenient to treat the no-arbitrage conditions in "physical units domain". The crucial observation is that the natural analog of the density processes of equivalent martingale measures in this general setting are strictly positive martingales evolving in the dual cone-valued process. The considered framework covers the majority of models considered in the literature, including the pioneering paper by Jouini and Kallal.

We discuss three types of no-arbitrage properties: weak (NA^w), strict (NA^s), and robust (NA^r), all coinciding with the classical one when the transaction cost coefficients are zero. The most natural generalization is the weak NA-property, claiming the absence of strict arbitrage opportunities, i.e., portfolio processes starting from zero and having as the terminal values a nontrivial random vector with positive coordinates. For finite probability space, the NA^w-criterion can be established in a very easy way, by the same arguments as the Harrison–Pliska theorem. Its formulation is simple: NA^w holds if and only if there exists a strictly positive martingale with values in the dual of solvency cones. Surprisingly, an extension to an arbitrary probability space, i.e., an analog of the Dalang–Morton–Willinger theorem, as it was shown by Schachermayer, fails to be true in general. By this reason other definitions of no-arbitrage were investigated by a number of authors. A particularly fruitful idea, due to Schachermayer, is to consider as arbitrage-free the models for which the NA^w-property still holds even under better investment opportunities, i.e., with "larger" solvency cones. Such a "robust" no-arbitrage property, referred to as NA^r, allows for an equivalent ("dual") description without any assumptions on the underlying probability space. We present another surprising result, due to Grigoriev: in the two-asset model the NA^w-criterion in the above formulation still holds.

Another interesting feature of models with transaction costs is the presence of arbitrage of the second kind. The latter notion serves to describe the situation that the initial endowments of the investor lie outside the solvency cone. Nevertheless, there is a self-financing portfolio which ends up in the solvency cone. It happens that the absence of arbitrage of the second kind is equivalent to the existence of martingales evolving in the interiors of dual of the solvency cones with arbitrary starting points.

The hedging problem for European contingent claims for markets with transaction costs can be formulated as follows. The contingent claim ξ is a vector of liabilities expressed in the units of corresponding assets. The investor wants to know whether he can super-replicate the contingent claim (in the sense of partial ordering generated by the solvency cone at the terminal date) by a self-financing portfolio starting from the initial endowment x. The answer is: the initial endowment x allows this if and only if its value Z_0x is not less than the expected value of the contingent claim $EZ_T\xi$ whatever is the process Z evolving in the dual to the solvency cone (a suggestive name for such a process Z: consistent price system).

The American contingent claim is a process. The option seller is interested to determine whether his initial endowment x suits to start a portfolio super-replication the American contingent claim at all dates. We present in Chap. 3 a hedging theorem which involves a class of coherent price systems which is larger than the class of consistent price systems.

We explain, following Bouchard, that models where the investor's information is delayed or restricted can be treated in the space of orders which is of higher dimension and give no-arbitrage criteria for such a situation.

We provide some results on the other important theoretical problem, hedging theorem in continuous-time framework. The latter gives a description of the set of initial vector-valued endowments ensuring the existence of a hedging portfolio for a given, also vector-valued, contingent claim. The situation here is more complicated than in the discrete-time setting: one needs an appropriate definition of admissibility and even that of portfolio processes. We present here the recent result due to Campi and Schachermayer explaining that the requirement that the portfolio process is càdlàg is too restrictive to get a "good" hedging theorem and should be replaced. We complete the chapter by a hedging theorem for American options.

The concluding chapter is devoted to Davis–Norman consumption–investment problem in a multi-asset framework. We start by recalling the classical Merton problem for a power utility function. From the point of view of experts in stochastic control the latter is trivial. Indeed, the verification theorem (which is itself a very simple result) requires to find a solution of the Hamilton–Jacobi–Bellman (HJB) equation. An easy argument shows that the Bellman function inherits the homogeneity property of the utility function and, thus, if finite, it is the same utility function up to a multiplicative constant, and the HJB equation is reduced immediately to an algebraic one to determine the latter.

The situation becomes quite different in the transaction cost setting. The problem is far from being trivial even in the case of powerful utility function. There is no smooth solution, and, therefore, the usual verification theorem does not work. The remedy comes from the theory of viscosity solutions. However, one should first check that the Bellman function is a viscosity solution of the HJB equation. Though the general lines of the arguments are well known, they are quite lengthy and rarely presented in detail. Moreover, there are many definitions of the viscosity solutions. We used the simplest one adapted for positive utility functions. In this chapter we give a rigorous proof of the Dynamic Programming Principle and derive that the Bellman function is a viscosity solution of the HJB equation. We proof that the latter has a unique solution in a class determined by a suitably defined Lyapunov function. Following Soner–Shreve, we analyze the structure of the Bellman function in the case of the two-asset model and conclude by presenting the results of Shreve and Janeček on the asymptotics of the solution when the transaction costs tend to zero.

In the Appendix we collect various auxiliary results from convex geometry, functional analysis, probability theory, measurable selection, and stochastic differential equations with reflection. Of course, our bibliographical comments are not exhaustive, and we apologize in advance for missing references.

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